Entanglement-assisted Enhanced Information Transmission Over a Quantum Channel with Correlated Noise; A General Expression

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Entanglement and entanglement-assisted are useful resources to enhance the mutual information of the Pauli channels, when the noise on consecutive uses of the channel has some partial correlations. In this paper, We study quantum communication channels with correlated noise and derive a general expression for the mutual information of quantum channel, for the product, maximally entangled state coding and entanglement-assisted systems with correlated noise in the Pauli quantum channels. Hence, we suggest more efficient coding in the entanglement-assisted systems for the transmission of classical information and derive a general expression for the entanglement-assisted classical capacity. Our results show that in the presence of memory, a higher amount of classical information is transmitted by two or four consecutive uses of entanglement-assisted systems.

I. INTRODUCTION

One of the remarkable byproducts of the development of quantum mechanics in recent years is quantum information and quantum computation theories. Classical and quantum information theories have some basic differences. Some of these differences are superposition principle, uncertainty principle and non-local effects. The non-locality associated with entanglement in quantum mechanics is one of the most subtle and intriguing phenomena in nature [1]. Its potential usefulness has been demonstrated in a variety of applications, such as quantum teleportation, quantum cryptography, and quantum dense coding. On the other hand, quantum entanglement is a fragile feature, which can be destroyed by interaction with the environment. This effect, which is due to decoherence [2], is the main obstacle for practical implementation of quantum computing and quantum communication. Several strategies have been devised against decoherence. Quantum error correction codes, fault-tolerant quantum computation [1] and dechorence free subspaces [3] are among them. One of the main problems in the quantum communication is the decoherence effects in the quantum channels.

Recently, the study of quantum channels has attracted a lot of attention. Early works in this direction were devoted, mainly, to memoryless channels for which consecutive signal transmissions through the channel are not correlated. The capacities of some of these channels were determined [4, 5] and it was proven that in most cases their capacities are additive for single uses of the channel. For Gaussian channels under Gaussian inputs, the multiplicativity of output purities was proven in [6] and the additivity of the energy-constrained capacity, even in the presence of classical noise and thermal noise, was proven in [7], under the assumption that successive uses of the channel are represented by the tensor product of the operators representing one single use of the channel, i.e., the channel is memoryless.

In a recent letter, Bartlett *et al.* [8] showed that it is possi-

ble to communicate with perfect fidelity, and without a shared reference frame, at a rate that asymptotically approaches one encoded qubit per transmitted qubit. They proposed a method to encode a qubit, using photons in a decoherence-free subspace of the collective noise model. Boileau *et al.* considered collective-noise channel effects in the quantum key distribution [9] and they gave a realistic roboust scheme for quantum communication, with polarized entangled photon pairs [10]. In the last few years much attention has been given to bosonic quantum channels [11].

Recently Macchiavello *et al.* [12, 13], considered a different class of channels, in which correlated noise acts on consecutive uses of channels. They showed that higher mutual information can be achieved above a certain memory threshold, by entangling two consecutive uses of the channel. This type of channels and its extension to the bosonic case, has attracted a lot of attention in the recent years [14].

K. Banaszek *et. al* [15] implemented Macchiavello *et al.* suggestion experimentally. They shown how entanglement can be used to enhance classical communication over a noisy channel. In their setting, the introduction of entanglement between two photons is required in order to maximize the amount of information that can be encoded in their joint polarization degree of freedom, and they obtained experimental classical capacity with entangled states and showed that it is more than 2.5 times the theoretical upper limit, when no quantum correlations are allowed. Hence, recently some people show that provided the sender and receiver share prior entanglement, a higher amount of classical information is transmitted over Pauli channels in the presence of memory, as compared to product and entangled state coding [16].

In this Paper, we show that if parties use a semi-quantum approach to entanglement-assisted coding, a higher amount of classical information is transmitted. We derive a general expression for the entanglement-assisted classical capacity and compare our results with the product Bell states and entanglement-assisted coding for various types of Pauli channels.

This Paper is organized as follows: In Sec. II we briefly review some properties of quantum memory channels and derive the general expression of classical capacity of quantum

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channel for product and maximally entangled state coding. In Sec. III, we derive the general expression of entanglement-assisted classical capacity of quantum channels with correlated noise that was calculated previously for Pauli channels by Arshed and Toor (AT). In Sec IV, we show that AT model is not an optimal coding for transmission of classical information, and we suggest another sets of states and derive the general expression of entanglement-assisted classical capacity. Our results show that a higher amount of classical capacity can indeed be achieved for all values of memory, by two or four consecutive uses of the entanglement-assisted systems for depolarizing, flip, two Pauli and phase damping channels.

II.   ENTANGLEMENT-ENHANCED INFORMATION TRANSMISSION OVER A QUANTUM CHANNEL WITH CORRELATED NOISE

Encoding classical information into quantum states of physical systems gives a physical implementation of the constructs of information theory. The majority of research into quantum communication channels has focused on the memoryless case, although there have been a number of important results obtained for quantum channels with correlated noise operators or more general quantum channels [12, 17].

The action of transmission channels is described by Kraus operators A_i [18], which satisfy the $\sum_i A_i^\dagger A_i \leq 1$, the equality holds when the map is trace-preserving. Thus, if we send a qubit in a state described by the density operator ρ, through the channel, then the corresponding out put state is given by the map.

$$\rho \longrightarrow \varepsilon(\rho) = \sum_i A_i \rho A_i^\dagger$$

An interesting class of Kraus operators acting on individual qubits can be expressed in terms of the Pauli operators $\sigma_{x,y,z}$

$$A_i = \sqrt{p_i}\sigma_i \ ,$$

with $\sum_i p_i = 1$, $i = 0, x, y, z$ and $\sigma_0 = I$. A noise model for these actions is, for instance, the application of a random rotation by angle π around the axis $\hat{x}, \hat{y}, \hat{z}$ with the probabilities p_x, p_y, p_z respectively, and the identity with probability p_0.

In the simplest scenario, the transmitter can send one qubit at a time along the channel. In this case the codewords will be restricted to the tensor products of the states of the individual qubits. Quantum mechanics, however allows also the possibility to entangle multiple uses of the channel.

Recently, a model for quantum channels with memory has been proposed that can consistently define quantum channels with Markovian correlated noise [12]. The model is also extended to describe channels that act on transmitted states in such a way that there is no requirement for interactions with an environment within the model. A Markovian correlated noise channel of length n, is of the form:

$$\varepsilon(\rho) = \sum_{i_1 \cdots i_n} p_{i_1} p_{i_2|i_1} \cdots p_{i_n|i_{n-1}} \qquad (1)$$
$$\times (A_{i_n} \otimes \cdots \otimes A_{i_1})\rho(A_{i_1}^\dagger \otimes \cdots \otimes A_{i_n}^\dagger)$$

where the A_{i_k} are Kraus operators for single uses of the channel on the state k and $p_{i_k|i_{k-1}}$ can be interpreted as the conditional probability that a π rotation around the axis i_k is applied to the k-th qubit, given that a π rotation around axis i_{k-1} was applied on the $k-1$-th qubit. They considered [12] the case of two consecutive uses of a channel with a partial memory, i.e. $p_{i_k|i_{k-1}} = (1-\mu)p_{i_k} + \mu\delta_{i_k|i_{k-1}}$. This means that with the probability μ, the same rotation is applied to both qubits, while with probability $1-\mu$ the two rotations are uncorrelated. Then, they concentrated their attention on the depolarizing channel, for which $p_0 = 1-p$ and $p_i = p/3, i = x, y, z$, and showed that for the specific case of a quantum depolarizing channel with collective noise, the transmission of classical information can be enhanced by employing maximally entangled states as carriers of information, rather than product states.

In this section, we would like derive a general expression for the classical capacity of Pauli quantum channels with correlated noise for Bell and product states. The maximum mutual information $I(\varepsilon(\rho))$ of a general quantum channel ε is given by the Holevo-Schumacher-Westmoreland bound [4]

$$C(\varepsilon) = Max_{[\pi_i, \rho_i]}S(\varepsilon(\sum_i \pi_i\rho_i)) - \sum_i \pi_i S(\varepsilon(\rho_i)) \quad (2)$$

where $S(\omega) = -Tr(\omega \log_2 \omega)$ is the von Neumann entropy of the density operator ω and the maximization is performed over all input ensembles π_i and ρ_i. Note that this bound incorporates maximization over all POVM (positive operator value measures) measurements at the receiver, including the collective ones over multiple uses of the channel.

In what follows, we shall derive $I(\varepsilon(\rho))$ for maximally Bell and product states. For the Bell states which are defined as:

$$|\Psi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle),$$
$$|\Phi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \qquad (3)$$

we know that the maximally entangled input states can be derived from each other by unitary transformations (for example $\rho_0 = |\Psi^+\rangle\langle\Psi^+|$), and with respect to the von Neumann entropy, $S(\omega)$ is invariant under any unitary transformation of a quantum state ω. The second term on the right hand side of equation (2) becomes:

$$\sum_i \pi_i S(\varepsilon(\rho_i)) = S(\varepsilon(\rho_0)) = -\sum_i \lambda_i \log_2 \lambda_i$$

where λ_i are eigenvalues of the transformed states. On the other hand, with respect to the relation $Tr_k \rho_i = \frac{1}{2}\mathbf{I}_l$ (with $k = A, B$ and $l = B, A$), we show that the Holevo limit can be attained by setting $\pi_i = \frac{1}{2^2}$ (with $i = 0, ..., 3$). The quantum state $\varepsilon(\rho_i)$, in the first term, can be writhen as:

$$\varepsilon(\sum_i \pi_i\rho_i) = \frac{1}{2^2}\sum_i \varepsilon(\rho_i) = \varepsilon(\frac{1}{2}\mathbf{I}_1 \otimes \frac{1}{2}\mathbf{I}_2)$$

To get the final result, we take suitable bases for the density matrixes representation. Then, the mutual information $I(\varepsilon(\rho))$ of quantum channel is given by:

$$I(\varepsilon(\rho)) = 2 + \sum_i \lambda_i \log_2 \lambda_i \qquad (4)$$

which for Bell states have the eigenvalues:

$$
\begin{aligned}
\lambda_1 &= (1-\mu)[(p_0^2 + p_x^2 + p_y^2 + p_z^2)] + \mu \qquad (5) \\
\lambda_2 &= (1-\mu)2(p_0 p_z + p_x p_y) \\
\lambda_3 &= (1-\mu)2(p_0 p_x + p_y p_z) \\
\lambda_4 &= (1-\mu)2(p_0 p_y + p_x p_z)
\end{aligned}
$$

With a similar approach, it is straightforward to verify that eigenvalues of product states in the z basis $|jk\rangle_z$ (with $j, k = 0, 1$) are give by:

$$
\begin{aligned}
\lambda_1^z &= (1-\mu)(p_0 + p_z)^2 + \mu(p_0 + p_z) \qquad (6) \\
\lambda_2^z &= (1-\mu)(p_x + p_y)^2 + \mu(p_x + p_y) \\
\lambda_{3,4}^z &= (1-\mu)(p_0 + p_z)(p_x + p_y)
\end{aligned}
$$

By inserting these eigenvalues in the eq. (4), the mutual information is give for the Bell and product states.

It is not complicated to verify that if we use product states in x basis $|jk\rangle_x$ (with $j, k = 0, 1$ and $|l\rangle_x = (|0\rangle + (-1)^l|\rangle)_z$, $l = j, k$), the eigenvalues in this basis are identical to the above eigenvalues with p_x and p_z being interchanged. Hence, if we use the product states in the y basis $|jk\rangle_y$ (with $j, k = 0, 1$ and $|l\rangle_y = (|0\rangle + i(-1)^l|\rangle)_z$, $l = j, k$), the eigenvalues in this basis are identical to above eigenvalues with, p_y and p_z being interchanged.

These eigenvalues reduce to previous amounts in the especial case of depolarizing channels [12]. In the final section, we compare these mutual information to the entanglement-assisted classical capacity.

III. ENTANGLEMENT-ASSISTED ENHANCED CLASSICAL CAPACITY OF QUANTUM CHANNELS WITH CORRELATED NOISE

Recently, Arshed and Toor [16] considered an interesting extension of classical capacity, and calculated the entanglement-assisted classical capacity of Pauli channels for two consecutive uses of the channels. They assumed that the sender (Alice) and the receiver (Bob) share two (same or different) maximally entangled Bell states $|\psi^\pm\rangle$, $|\phi^\pm\rangle$ (as defined by the eq. (3)). The first qubits of Bell states belongs to Alice while the second qubits belongs to Bob. Alice sends her qubits through the channel. They calculated eigenvalues of the pure density matrix, transformed under the action of the depolarizing channel, for probabilities $p_0 = 1 - p$, $p_x = p_y = p_z = p/3$, and after some simple calculations, they derived the entanglement-assisted classical capacity for the Pauli channels in the presence of partial memory.

As was suggested by AT, in the presence of partial memory, the action of Pauli channels is described by the Kraus operators:

$$A_{i,j}(\mu) = \sqrt{p_i[(1-\mu)p_j + \mu\delta_{i,j}]}(\sigma_i^A \otimes I^B) \otimes (\sigma_j^A \otimes I^B)$$

The mutual information of quantum dense coding system, where Alice and Bob shared quantum state ρ^{AB} (which is statistical mixture of the Bell states), is given by:

$$I^{AT}(\varepsilon(\rho)) = 4 + \sum_i \lambda_i \log_2 \lambda_i \qquad (7)$$

where λ_i are the eigenvalues of the transformed states. For states that were considered by AT, eigenvalues in the general case are give by:

$$
\begin{aligned}
\lambda_1 &= (1-\mu)p_0^2 + \mu p_0 \qquad (8) \\
\lambda_2 &= (1-\mu)p_x^2 + \mu p_x \\
\lambda_3 &= (1-\mu)p_y^2 + \mu p_y \\
\lambda_4 &= (1-\mu)p_z^2 + \mu p_z \\
\lambda_{5,6} &= (1-\mu)p_0 p_x \\
\lambda_{7,8} &= (1-\mu)p_0 p_y \\
\lambda_{9,10} &= (1-\mu)p_0 p_z \\
\lambda_{11,12} &= (1-\mu)p_x p_y \\
\lambda_{13,14} &= (1-\mu)p_x p_z \\
\lambda_{15,16} &= (1-\mu)p_y p_z
\end{aligned}
$$

These eigenvalues reduce those of AT approach in the especial cases of depolarizing, flip, two-Pauli and phase damping channels [16].

IV. SEMI-ENTANGLEMENT-ASSISTED ENHANCED INFORMATION TRANSMISSION OVER QUANTUM CHANNELS WITH CORRELATED NOISE

In the preceding section, we derived the classical information capacity $C(\varepsilon)$ of a quantum dense coding system. Although, the use of entanglement-assisted enhances classical capacity of channels with correlated noise, in comparison with both product and entangled state coding for all values of μ, here exists other quantum dense coding systems in which those states have a higher amount of classical information transmission in the channels with correlated noise.

In this section, we derive a general expression for the entanglement-assisted classical capacity for two kinds of quantum dense coding systems which we call semi-quantum dense coding. Our results show that for channels with partial memory and with the use of appropriate choice of entanglement-assisted states, higher mutual information can indeed be achieved for all values of μ, by two or four consecutive uses of the channel. First, we consider a simple modification of AT approach. As we saw, AT use two maximally entangled Bell states, the first qubit belongs to the Alice and the second qubit belongs to the Bob, we replace AT states by

following states:

$$|\Phi_{1,2}\rangle = |\psi^+\rangle_A|\psi^+\rangle_B \pm |\psi^-\rangle_A|\psi^-\rangle_B \qquad (9)$$

$$|\Phi_{3,4}\rangle = |\psi^-\rangle_A|\psi^+\rangle_B \pm |\psi^+\rangle_A|\psi^-\rangle_B$$

$$|\Phi_{5,6}\rangle = |\varphi^+\rangle_A|\psi^+\rangle_B \pm |\varphi^-\rangle_A|\psi^-\rangle_B$$

$$|\Phi_{7,8}\rangle = |\varphi^-\rangle_A|\psi^+\rangle_B \pm |\varphi^+\rangle_A|\psi^-\rangle_B$$

$$|\Phi_{9,10}\rangle = |\psi^+\rangle_A|\varphi^+\rangle_B \pm |\psi^-\rangle_A|\varphi^-\rangle_B$$

$$|\Phi_{11,12}\rangle = |\psi^-\rangle_A|\varphi^+\rangle_B \pm |\psi^+\rangle_A|\varphi^-\rangle_B$$

$$|\Phi_{13,14}\rangle = |\varphi^+\rangle_A|\varphi^+\rangle_B \pm |\varphi^-\rangle_A|\varphi^-\rangle_B$$

$$|\Phi_{15,16}\rangle = |\varphi^-\rangle_A|\varphi^+\rangle_B \pm |\varphi^+\rangle_A|\varphi^-\rangle_B$$

In these states, the first Bell states belongs to Alice and the second Bell states belongs to Bob, which are represented by subscriptions in the Bell states respectively. These states are equivalent to the Bell state which is shared by Alice and Bob. For example, parties can construct $|\Phi_1\rangle = |00\rangle_A|00\rangle_B + |11\rangle_A|11\rangle_B$ by performing CNOT gates C_{A1} and C_{B2} on the state $(|00\rangle_{AB} + |11\rangle_{AB})|0\rangle_1|0\rangle_2$ (A and B are the controller and 1 and 2 are the targets). Hence, Alice can transform $|\Phi_i\rangle$ (with $i = 1, ..., 8$) states to each other by a local operation on the his qubits. The same transformation exists for $|\Phi_i\rangle$ (with $i = 9, ..., 16$). These two groups of states cannot transform to each other by Alice's local operation.

On the other hand, as we know, if Alice and Bob previously shared entangled states between themselves, then, for noiseless quantum channels, the amount of classical information transmitted through a quantum channel is doubled in comparison with the unshared states $C_E = 2C$ [19]. In other words, every previously shared Bell state is equivalent with two quantum channels (or twice the using quantum channel).

In our approach, Alice and Bob shared only one of the above states (for example, $|\Phi_1\rangle$) and they had previously com-

promised that if Alice states were received by Bob in the time interval $\delta t_0 = t_1 - t_0$ ($\delta t_1 = t_2 - t_1$), Bob would operate $I^B \otimes I^B$ ($I^B \otimes \sigma_x^B$) on the his qubits (the Bell states that were represented by subscript 2 in the (9)) [20].

By using this protocol, Alice and Bob get access to all Hilbert space states in the states (9). This protocol (using one shared Bell state between parties and two types of qubits transmission), we call semi-quantum dense coding which has the same cost as that of the AT approach (which uses two shared Bell states between parties and one type of qubits transmission in the quantum channel).

We consider the shortest variational time of the channel (fiber) under thermal and mechanical fluctuations as τ_{fluc}. If the time lapse (τ_{lap}) between the two used channel is small compared to τ_{fluc}, the effects of the channel on various qubits can be considered as a correlated noise. For example, in the experiment of K. Banaszek $et.$ al [15] $\tau_{lap} \approx 6ns$, which is much smaller than the mechanical fluctuations of the fiber.

Similar to the previous cases, the action of transmission channels is described by Karus operators $A_{i,j}$, satisfying $\sum_{i,j} A_{i,j} A_{i,j}^\dagger = 1$, Where in the presence of partial memory, the action of Pauli channels on ρ is described by

$$A_{i,j}(\mu) = \sqrt{p_i[(1-\mu)p_j + \mu\delta_{i,j}]}(\sigma_i \otimes \sigma_j) \otimes (I \otimes I)$$

If Alice sends the density matrix ρ through the quantum channel, the corresponding output state is given by the mapping

$$\rho \longrightarrow \varepsilon(\rho) = \sum_{i,j=0}^{3} A_{i,j}(\mu)\rho A_{i,j}^\dagger(\mu) \qquad (10)$$

That can be written for channels with partial memory as:

$$\varepsilon(\rho) = (1-\mu)\sum_{i,j=0}^{3} p_i p_j(\sigma_i \otimes \sigma_j \otimes I \otimes I)\rho(\sigma_i \otimes \sigma_j \otimes I \otimes I) + \mu\sum_{i=0}^{3} p_i(\sigma_i \otimes \sigma_i \otimes I \otimes I)\rho(\sigma_i \otimes \sigma_i \otimes I \otimes I)$$

Here, we would like to compute the entanglement-assisted classical capacity of quantum channels by using quantum superdense coding system, where Alice and Bob share a completely entangled state $|\Phi_1^{AB}\rangle$, which belongs to an $(2^2 \times 2^2)$-dimensional (for 2^2-dimensional Bell states) Hilbert space $H \otimes H$, and the encoded system is sent through a quantum channel, described by an arbitrary trace-preserving completely positive map ε.

In a simple modification of Ban et $al.$ [21] approach to the calculation of classical capacity of quantum dense coding, we assume Alice and Bob to share a statistical mixture of the entangled states $|\Phi_i\rangle$ (with $i = 1, ..., 16$), where the shared quantum state ρ^{AB} is given by:

$$\rho^{AB} = \sum_{j=0}^{16} \lambda_j |\Phi_j^{AB}\rangle\langle\Phi_j^{AB}| \qquad (11)$$

with $\lambda_j \geq 0$ and $\sum_{j=0}^{16} \lambda_j = 1$. To encode $2\log_2 2^2$ bits of classical information, Alice applies one of 8×2 operation, 8 unitary operators V_k ($k = 0, ..., 8$) to her part of the quantum state ρ^{AB} and the two types of state transmission. She sends the encoded system to Bob through an arbitrary quantum channel ε. Then, Bob obtains the quantum state:

$$\rho'^{AB}_{jk} = (\varepsilon^A \otimes I^B)[(V_j^A \otimes v_k^B)\rho^{AB}(V_j^{A\dagger} \otimes v_k^{B\dagger})] \qquad (12)$$

In the above relation, v_k^B depends on the type of qubits sent

by Alice, which operate by Bob on the his qubits. If the prior probability of the classical information corresponding to V_j^A and v_k^B (Alice and Bob operation on the shared state) is π_{jk}, where $\pi_{jk} \geq 0$ and $\sum_{j=0}^{8} \sum_{k=0}^{1} \pi_{jk} = 1$, the maximum amount of mutual information of the quantum dense coding system would be given by [22]:

$$
\begin{aligned}
C^E(\varepsilon, \pi_{jk}) &= Max_{[\rho, \pi_{jk}]} I^{AB}(\varepsilon, \pi_{jk}) \qquad (13) \\
&= Max_{[\rho, \pi_{jk}]} S(\rho'^{AB}) - \sum_{j,k} \pi_{jk} S(\varepsilon(\rho_{jk}))
\end{aligned}
$$

where $\rho'^{AB} = \sum_j \sum_k \pi_{jk} \varepsilon(\rho_{jk}^{AB})$. In the above relation the maximization is performed over all input ensembles π_{jk} and ρ_{ik}. Note that this bound incorporates maximization over all POVM measurements (positive operator value measures) at the receiver, including the collective ones over multiple uses of the channel. In what follows, we shall derive $I(\varepsilon(\rho_{jk}))$ for entangled states which were suggested in the eq.(9). We know that the entangled input states $|\Phi_i\rangle$ can be derived from $|\Phi_1\rangle$ by unitary transformations (Pauli matrixes), and with respect to the von Neumann entropy, $S(\omega)$ is invariant under any unitary transformation of a quantum state ω. The second term on the right hand side of equation (13) becomes:

$$
\sum_{j,k} \pi_{jk} S(\varepsilon(\rho_{jk})) = S(\varepsilon^A \otimes I^B(\rho^{AB}))
$$

On the other hand, we show that the maximum amount of mutual information can be attained by setting $\pi_{jk} = \frac{1}{2^4}$ (with $j = 0, ..., 8$ and $k = 0, 1$). The quantum state $\varepsilon(\rho_{jk}^{AB})$, in the first term, can be written as:

$$
\sum_{jk} \pi_{jk} \varepsilon(\rho_{jk}^{AB}) = \frac{1}{2^4} \sum_{jk} \varepsilon(\rho_{jk}^{AB}) = \varepsilon(\frac{1}{2}\mathbf{I}_1 \otimes ... \otimes \frac{1}{2}\mathbf{I}_4)
$$

To get the final result, we take suitable bases for density matrixes representation. Mutual information $I_1^{AB,e-a}(\varepsilon, \pi_{jk})$ can be calculated for the quantum channel with entangled states (9) in the general case. Therefore, for symmetric and

asymmetric Pauli channels, we have:

$$
I_1^{AB,e-a}(\varepsilon, \pi_{jk}) = 4 - S(\varepsilon^A \otimes I^B(\rho^{AB})) \qquad (14)
$$

The von Neumann entropy of the ρ^{AB} state, transformed under the action of Pauli channels, is given by:

$$
S(\varepsilon^A \otimes I^B(\rho^{AB})) = -\sum_i \lambda_i \log_2 \lambda_i \qquad (15)
$$

where λ_i are the eigenvalues of the transformed ρ^{AB} state, which have the explicit forms:

$$
\begin{aligned}
\lambda_1 &= (1-\mu)(p_0^2 + p_z^2) + \mu(p_0 + p_z) \qquad (16) \\
\lambda_2 &= (1-\mu)(p_x^2 + p_y^2) + \mu(p_x + p_y) \\
\lambda_3 &= (1-\mu)2p_0 p_z \\
\lambda_4 &= (1-\mu)2p_x p_y \\
\lambda_{5,6} &= (1-\mu)(p_0 p_x + p_y p_z) \\
\lambda_{7,8} &= (1-\mu)(p_0 p_y + p_x p_z)
\end{aligned}
$$

Hence, there exist two groups of eigenvalues which are equal to each other. These states don't mix with each other through interaction with environment.

Although the above states are more efficient than the AT states (as we shall see at the end of this section) for the transmission of classical information, there exist other entangled states which enhance the classical capacity of quantum channels through the use of the entanglement-assisted. Similar to the previous case, we consider semi-quantum approach, which uses one entangled state augmented by quantum channels and show that in the channels with partial memory, the use of four particle entanglement-assisted enhances the amount of mutual information can indeed be achieved for all values of μ, by four consecutive uses of the channel (at more of cases, for example, in the depolarizing, flip and two Pauli channels). This approach has 8^4 dimensional Hilbert space, that in following we represent sixty four (one quarter of total Hilbert space dimension) of them:

$$
\begin{aligned}
|\Psi_{1,i}\rangle &= |\psi^+\rangle_1|\psi^+\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 + |\psi^-\rangle_1|\psi^-\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 + |\phi^+\rangle_1|\phi^+\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 + |\phi^-\rangle_1|\phi^-\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4 \quad (17) \\
|\Psi_{2,i}\rangle &= |\psi^+\rangle_1|\psi^+\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 + |\psi^-\rangle_1|\psi^-\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 - |\phi^+\rangle_1|\phi^+\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 - |\phi^-\rangle_1|\phi^-\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4 \\
|\Psi_{3,i}\rangle &= |\psi^+\rangle_1|\psi^+\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 - |\psi^-\rangle_1|\psi^-\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 - |\phi^+\rangle_1|\phi^+\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 + |\phi^-\rangle_1|\phi^-\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4 \\
|\Psi_{4,i}\rangle &= |\psi^+\rangle_1|\psi^+\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 - |\psi^-\rangle_1|\psi^-\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 + |\phi^+\rangle_1|\phi^+\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 - |\phi^-\rangle_1|\phi^-\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4 \\
|\Psi_{5,i}\rangle &= |\psi^+\rangle_1|\psi^-\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 + |\psi^-\rangle_1|\psi^+\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 + |\phi^+\rangle_1|\phi^-\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 + |\phi^-\rangle_1|\phi^+\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4 \\
|\Psi_{6,i}\rangle &= |\psi^+\rangle_1|\psi^-\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 + |\psi^-\rangle_1|\psi^+\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 - |\phi^+\rangle_1|\phi^-\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 - |\phi^-\rangle_1|\phi^+\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4 \\
|\Psi_{7,i}\rangle &= |\psi^+\rangle_1|\psi^-\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 - |\psi^-\rangle_1|\psi^+\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 - |\phi^+\rangle_1|\phi^-\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 + |\phi^-\rangle_1|\phi^+\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4 \\
|\Psi_{8,i}\rangle &= |\psi^+\rangle_1|\psi^-\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 - |\psi^-\rangle_1|\psi^+\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 + |\phi^+\rangle_1|\phi^-\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 - |\phi^-\rangle_1|\phi^+\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4 \\
|\Psi_{9,i}\rangle &= |\psi^+\rangle_1|\phi^+\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 + |\psi^-\rangle_1|\phi^-\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 + |\phi^+\rangle_1|\psi^+\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 + |\phi^-\rangle_1|\psi^-\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4 \\
|\Psi_{10,i}\rangle &= |\psi^+\rangle_1|\phi^+\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 + |\psi^-\rangle_1|\phi^-\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 - |\phi^+\rangle_1|\psi^+\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 - |\phi^-\rangle_1|\psi^-\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4
\end{aligned}
$$

$$|\Psi_{11,i}\rangle = |\psi^+\rangle_1|\phi^+\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 - |\psi^-\rangle_1|\phi^-\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 - |\phi^+\rangle_1|\psi^+\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 + |\phi^-\rangle_1|\psi^-\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4$$

$$|\Psi_{12,i}\rangle = |\psi^+\rangle_1|\phi^+\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 - |\psi^-\rangle_1|\phi^-\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 + |\phi^+\rangle_1|\psi^+\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 - |\phi^-\rangle_1|\psi^-\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4$$

$$|\Psi_{13,i}\rangle = |\psi^+\rangle_1|\phi^-\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 + |\psi^-\rangle_1|\phi^+\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 + |\phi^+\rangle_1|\psi^-\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 + |\phi^-\rangle_1|\psi^+\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4$$

$$|\Psi_{14,i}\rangle = |\psi^+\rangle_1|\phi^-\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 + |\psi^-\rangle_1|\phi^+\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 - |\phi^+\rangle_1|\psi^-\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 - |\phi^-\rangle_1|\psi^+\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4$$

$$|\Psi_{15,i}\rangle = |\psi^+\rangle_1|\phi^-\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 - |\psi^-\rangle_1|\phi^+\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 - |\phi^+\rangle_1|\psi^-\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 + |\phi^-\rangle_1|\psi^+\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4$$

$$|\Psi_{16,i}\rangle = |\psi^+\rangle_1|\phi^-\rangle_2|\omega^i\rangle_3|\psi^+\rangle_4 - |\psi^-\rangle_1|\phi^+\rangle_2|\omega^{i+1}\rangle_3|\psi^-\rangle_4 + |\phi^+\rangle_1|\psi^-\rangle_2|\omega^{i+2}\rangle_3|\phi^+\rangle_4 - |\phi^-\rangle_1|\psi^+\rangle_2|\omega^{i+3}\rangle_3|\phi^-\rangle_4$$

In the above states, $i = 0, ..., 3$ and $|\omega^i\rangle$ are define as:

$$|\omega^{0,1}\rangle = |\Psi^\pm\rangle \qquad |\omega^{2,3}\rangle = |\Phi^\pm\rangle$$

the subscript i is calculated in the $mod\ 4$. In the above states, the first two Bell states of $1, 2$ belongs to Alice and the second two Bell states of $3, 4$ belongs to Bob which are represented by subscriptions in the Bell states respectively.

Similar to the previous case, Alice and Bob share only one of the above states (for example, $|\Psi_{1,0}\rangle$) and Alice can transform this state to one of the above states (one quarter of the total Hilbert space dimension) by local operation on the her qubits.

They had previously compromised that if Alice states received by Bob were in the time intervals $\delta t_0 = t_1 - t_0$ or $\delta t_1 = t_2 - t_1$ or $\delta t_2 = t_2 - t_3$ or $\delta t_3 = t_4 - t_3$, Bob would operate $I^B \otimes I^B$ or $\sigma_z^B \otimes I^B$ or $I^B \otimes \sigma_x^B$ or $\sigma_z^B \otimes \sigma_x^B$ on his qubits respectively [23]. By using this protocol, Alice and Bob get access to all of the Hilbert space states.

Similar to the previous cases, Karus operators $A_{i,j,k,l}$, satisfy $\sum_{i,j,k,l} A_{i,j,k,l}A_{i,j,k,l}^\dagger = 1$, and in the presence of partial memory, the action of Pauli channels on the ρ is described by:

$$A_{i,j,k,l}(\mu) = \sqrt{p_i[(1-\mu)p_j p_k p_l + \mu\delta_{i,j}\delta_{j,k}\delta_{k,l}]}\sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \sigma_l \otimes I^{\otimes 4}$$

For simplicity, we consider only two types of sending particles: i) The two Bell states in the Alice's hand (subscribed by $1, 2$ in the states (17)) are sent at the same time ($\tau_{lap} \ll \tau_{fluc}$ for each pair). ii) The two Bell states are sent with a time delay ($\tau_{fluc} \ll \tau_{lap}$ for each pair). Although, we can consider the general case of the transmission of quantum states, but its explicit form is very complicated and doesn't clarify any physical properties. With a similar approach to the one we described in the previous case, the mutual information for the entanglement-assisted systems can be calculated. If Alice sends the density matrix ρ through a quantum channel, the corresponding output state is given by the map:

$$\rho \longrightarrow \varepsilon(\rho) = \sum_{i,j,k,l=0}^{3} A_{i,j,k,l}(\mu)\rho A_{i,j,k,l}^\dagger(\mu)$$

For channels with partial memory output, the density matrix can be written as following:

$$\varepsilon(\rho) = (1-\mu)\sum_{i,j,k,l=0}^{3} p_i p_j p_k p_l(\sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \sigma_l \otimes I^{\otimes 4})\rho(\sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \sigma_l \otimes I^{\otimes 4})$$
$$+\mu\sum_{i=0}^{3} p_i(\sigma_i \otimes \sigma_i \otimes \sigma_i \otimes \sigma_i \otimes I^{\otimes 4})\rho(\sigma_i \otimes \sigma_i \otimes \sigma_i \otimes \sigma_i \otimes I^{\otimes 4})$$

After a similar calculation to the previous case, we derive the mutual information of the quantum channel, with correlated noise, given by:

$$I_2^{AB,e-a}(\varepsilon, \pi_i) = 8 - S(\varepsilon^A \otimes I^B(\rho^{AB})) \qquad (18)$$

The von Neumann entropy of the ρ^{AB} state, transformed un-

der the action of Pauli channels, is given by:

$$S(\varepsilon^A \otimes I^B(\rho^{AB})) = -\sum_i \lambda_i \log_2 \lambda_i \qquad (19)$$

where λ_i are the eigenvalues of the transformed ρ^{AB} state with the explicit forms:

$$\begin{aligned}
\lambda_1 &= (1-\mu)(p_0^4 + p_z^4 + p_x^4 + p_y^4) + \mu \\
\lambda_{2,3,4} &= (1-\mu)[2(p_0 p_z)^2 + 2(p_x p_y)^2] \\
\lambda_{4,6,7} &= (1-\mu)[2(p_0 p_x)^2 + 2(p_y p_z)^2] \\
\lambda_{8,9,10} &= (1-\mu)[2(p_0 p_y)^2 + 2(p_x p_z)^2] \\
\lambda_{11,\ldots,14} &= (1-\mu)[p_0 p_z(p_0^2 + p_z^2) + p_x p_y(p_x^2 + p_y^2)] \\
\lambda_{15,\ldots,26} &= (1-\mu)[p_0 p_z(p_x^2 + p_y^2) + p_x p_y(p_0^2 + p_z^2)] \\
\lambda_{27,\ldots,30} &= (1-\mu)[p_0 p_x(p_0^2 + p_x^2) + p_y p_z(p_y^2 + p_z^2)] \\
\lambda_{31,\ldots,38} &= (1-\mu)[p_0 p_x(p_y^2 + p_z^2) + p_y p_z(p_0^2 + p_x^2)] \\
\lambda_{39,\ldots,42} &= (1-\mu)[p_0 p_y(p_x^2 + p_z^2) + p_x p_z(p_0^2 + p_y^2)] \\
\lambda_{43,\ldots,46} &= (1-\mu)[p_0 p_y(p_0^2 + p_y^2) + p_x p_z(p_x^2 + p_z^2)] \\
\lambda_{47,\ldots,52} &= (1-\mu)[4 p_0 p_x p_y p_z] \\
\lambda_{53,\ldots,60} &= (1-\mu)[(p_0 p_z + p_x p_y)(p_0 p_x + p_y p_z)] \\
\lambda_{61,\ldots,64} &= (1-\mu)[(p_0 p_z + p_x p_y)(p_0 p_y + p_x p_z)]
\end{aligned}$$

Here we have four sets of equal eigenvalues (corresponding to each four sets of states, in which every set has 64 elements). These states don't mix with each other through the interaction with environment.

In the following, we would like to consider some examples of Pauli channels, both symmetric and asymmetric, and to work out their entanglement-assisted classical capacity. For depolarizing channels which are symmetric types of Pauli channel, for which $p_0 = 1 - p$ and $p_i = p/3, i = x, y, z$. In Fig.(1), we plot the mutual information of quantum channel $I^{AB}(\varepsilon(\rho))$ and the mutual information corresponding to entanglement-assisted dense coding system $I^{AB,e-a}(\varepsilon(\rho))$ of the depolarization channel versus its memory coefficient μ. Fig.(1) shows that the use of semi-quantum approach enhances mutual information of the quantum channel, if one prior entanglement is shared by Alice and Bob. That has better capacity over both products, entangled states and entanglement-assisted coding for all values of μ. Similar to earlier works [12], there exists another memory threshold for two semi-quantum approaches. Explicit form of the memory threshold is a very complicated relation, but numerical calculation is shown in the Fig.(2) for $0 < p_i < 1/3$.

In Fig.(2) we see that as expected in the high error channels, $p_0 = p_i = p/3, i = x, y, z$, the memory threshold is equal to zero, i.e. $\mu_t = 0$. In other words, in the channels with high errors, any output density matrix can be transformed to the following form:

$$\varepsilon(\rho) = (1-\mu)\frac{1}{a^b} I^{\otimes b} + \mu \sigma \qquad (20)$$

With $a = b = 4$ for states (17) and $a = 4, b = 2$ for states (9). Hence, $Tr\sigma = 1, Tr\varepsilon(\rho) = 1$. Optimal mutual information is
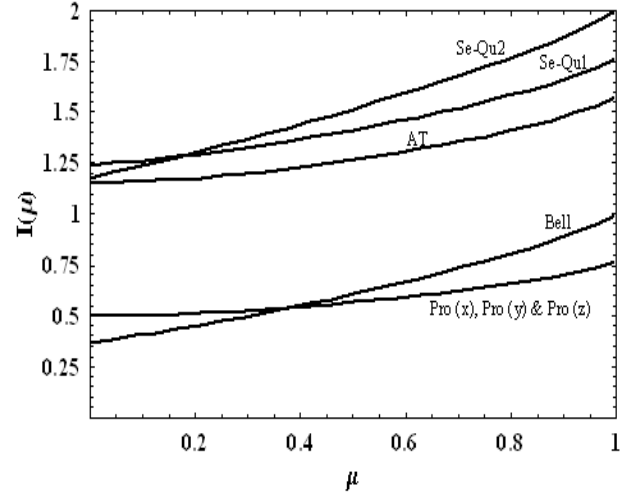

FIG. 1: Mutual information $I(\mu)$ for product state in the x basis (Pro(x)), y basis (Pro(y)) and z basis (Pro(z)), maximally entanglement state (Bell), entanglement-assisted (AT), first semi-quantum approach (Se-Qu1) and second semi-quantum approach (Se-Qu2) versus the memory coefficient μ for depolarizing channel, with $p_0 = 0.85$ and $p_i = 0.05, i = x, y, z$. The capacities are normalized with respect to the number of channel uses. Value of memory threshold of (Se-Qu1) and (Se-Qu2) for our choice of the probability of errors channel is $\mu_t = 0.171$.
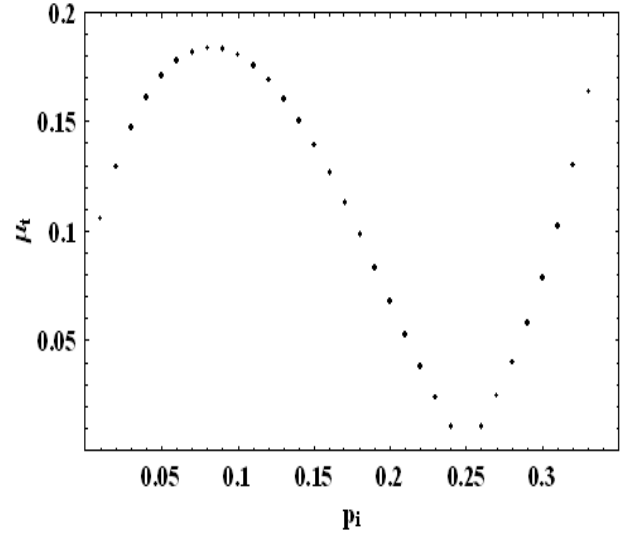

FIG. 2: Memory threshold μ_t as a function of the probability of errors p_i, $i = x, y, z$ in the depolarizing channel for first semi-quantum approach (Se-Qu1) and second semi-quantum approach (Se-Qu2).

obtained by minimizing the output entropy, and, for this, we must have a pure state at the output channel. The optimization of the the mutual information can be achieved by going to an appropriate bases that diagonaliyez σ. If we assume that σ has k none-zero diagonal elements, then, the entropy is given by:

$$S(\varepsilon(\rho)) = k\{(1-\mu)a^{-b} + \mu\frac{1}{k}\} \log_2\{(1-\mu)a^{-b} + \mu\frac{1}{k}\}$$
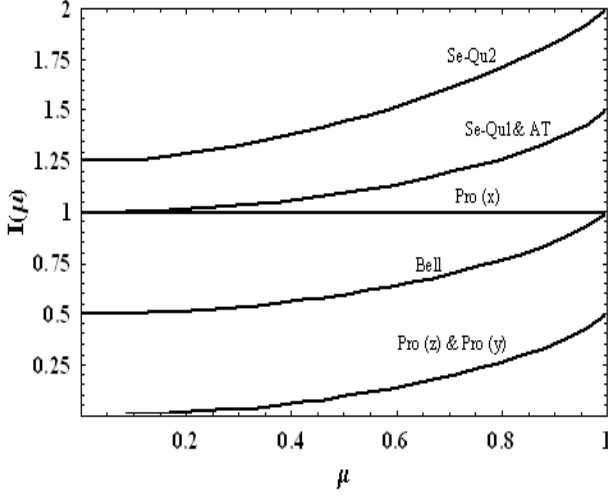
FIG. 3: Mutual information $I(\mu)$ for the product state in the x basis (Pro(x)), y basis (Pro(y)) and z basis (Pro(z)), maximally entanglement state (Bell), entanglement-assisted (AT), first semi-quantum approach (Se-Qu1) and second semi-quantum approach (Se-Qu2) versus the memory coefficient μ for flip channel (bit flip), with $p_0 = 0.5$ and $p_x = 0.5$. The capacities are normalized with respect to the number of channel uses.

$$+(a^b - k)\{(1 - \mu)a^{-b}\}\log_2\{1 - \mu)a^{-b}\}$$

Minimum value of the above relation can be obtained for $k = 1$. In the other words, σ must be a pure state, and this happens for the input states of (17).

On other hand, the maximum of memory threshold is give by $\mu_t^{Max} = 0.185$. Thus, for channels with $\mu \geq \mu_t^{Max}$, the use of (Se-Qu2) approach is more efficient.

In the following, we consider some examples of asymmetric Pauli channels [1]. The noise introduced by them is of three types: namely, bit flip, phase flip and bit-phase flip. Probability distribution for flip channels is given by $i, j = 0, f$, with probabilities $p_0 = 1 - p$ and $p_f = p$. Here $f = x, y$ and z for bit flip, phase flip and bit-phase flip channels, respectively. In Fig.(3), we plot the mutual information of $I^{AB}(\varepsilon(\rho))$ and the mutual information corresponding to entanglement-assisted dense coding system $I^{AB,e-a}(\varepsilon(\rho))$ of the bit flip channel versus its memory coefficient μ. Fig.(3) shows that the use of four particles, semi-quantum dense coding, approach enhances the capacity of quantum channel and the usual entanglement-assisted coding and two particle semi-quantum dense coding have the same plot for all values of μ. This shows that in the asymmetric Pauli channels, optimality in the unshared entangled states strictly depends on the noise of the channel. For example, in the bit flip channels, it is more appropriate to use x basis.

At another stage, we consider the two-Pauli channels [1]. The probability distribution for the two-Pauli channels is given by $i, j = 0, x, y$, with probabilities $p_0 = (1 - p)$ and $p_x = p_y = p/2$. In Fig.(4), we plot the mutual information of $I^{AB}(\varepsilon(\rho))$ and the mutual information corresponding to entanglement-assisted dense coding system $I^{AB,e-a}(\varepsilon(\rho))$ of the two-Pauli channels, versus its memory coefficient μ.
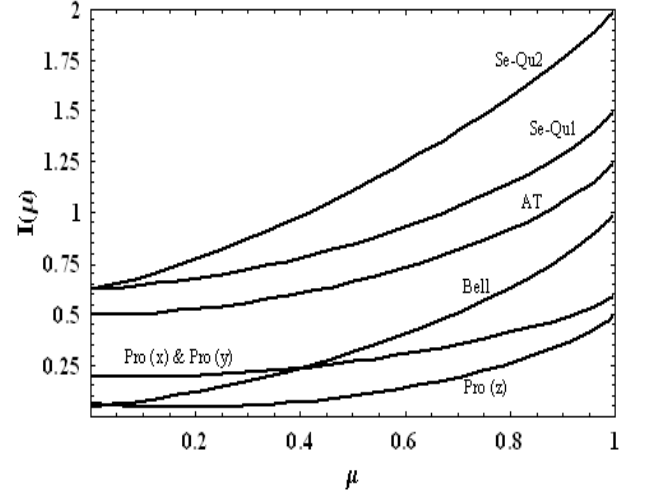

FIG. 4: Mutual information $I(\mu)$ for product state in the x basis (Pro(x)), y basis (Pro(y)) and z basis (Pro(z)), maximally entanglement state (Bell), entanglement-assisted (AT), first semi-quantum approach (Se-Qu1) and second semi-quantum approach (Se-Qu2) versus the memory coefficient μ for two-Pauli channel, with $p_0 = 0.5$ and $p_i = 0.25$, $i = x, y$. The capacities are normalized with respect to the number of channel uses. Value of memory threshold of (Pro(x),Pro(y)) state and (Bell) for our choice of probability of errors channel is $\mu_t = 0.409$.
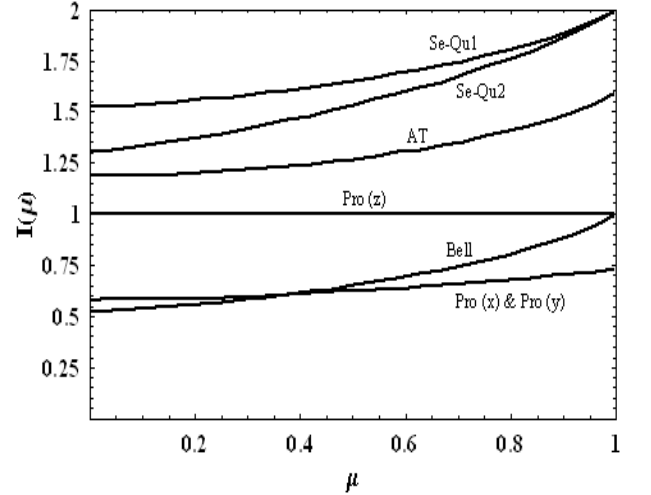

FIG. 5: Mutual information $I(\mu)$ for product state in the x basis (Pro(x)), y basis (Pro(y)) and z basis (Pro(z)), maximally entanglement state (Bell), entanglement-assisted (AT), first semi-quantum approach (Se-Qu1) and second semi-quantum approach (Se-Qu2) versus the memory coefficient μ for phase damping channel, with $p_0 = 0.75$ and $p_z = 0.25$. The capacities are normalized with respect to the number of channel uses.

This shows that the use of four particles semi-quantum dense coding approach enhances the capacity of quantum channel for all values of μ. Furthermore, at the usual stage (unshared state) [12], use of product states in the x or y basis would be more efficient than in the z basis and there exists a memory threshold where a higher amount of classical information is

transmitted with entangled states. Explicit form of memory threshold is very complicated, the numerical value of memory threshold for the above error model is $\mu_t = 0.409$.

Finally, we consider another type of asymmetric Pauli channels, the so-called phase damping channel [1]. The probability distribution for phase damping channels are given by $i, j = 0, z$ with probabilities $p_0 = (1 - p/2)$ and $p_z = p/2$. Fig.(5) shows that for phase damping channels, the use of two particles semi-quantum dense coding is more appropriate, compared with the four particle approach to information transmission, for all of values of μ. This shows that in the usual stage (previously unshared state) [12], the use of product states in the z basis would be more efficient than in the x or y basis.

On the other hand, we show that $I_n(\mathcal{E})$ is superadditive in the presence of entanglement-assisted, i.e. we have $I_{n+m}^E > I_n^E + I_m^E$ and therefore $C_n^E > C_1^E$. At this stage, the classical capacity C^E of the channel is defined by:

$$C^E = \lim_{n \to \infty} C_n^E .$$

It has been shown (similar to what we plotted in the figures) that the amount of reliable information which can be transmitted per use of the channel, is given by [4]:

$$C_n^E = \frac{1}{n} \sup_{\mathcal{E}} I_n^E(\mathcal{E}) ,$$

One of the main applications of the above extension of memorial channels is the extension of the standard quantum cryptography BB84 [24] to protocols where the key is carried by quantum states in a space of higher dimension, using two (or $d + 1$) mutually unbiased bases, which for high memorial channels have very low error rates. This procedure ensures that any attempt by any eavesdropper Eve to gain information on sender's state induces errors in the transmission, which can be detected by the legitimate parties [25]. On the other hand, if we are interested in other quantum key distribution protocols (such as the EPR protocol [26]), we must encode a qubit in a decoherence-free subspace of the collective noise for key distribution [9]. If we are interested in the total dimension of Hilbert space, we must revise the EPR protocol for this new approach. Another application of the above extension can be quantum coding and quantum superdence coding at higher dimensions. The errors in the memorial channels can be considered as a subset of collective noise which are considered in the decoherence-free subspace approach. Some experimental results [15] show that in some special cases the use of these states are appropriate, because in the memorial channels we make use of all maximally entangled states.

In conclusion, we have calculated a general expression for the classical capacity of a quantum channel for product states, maximally entangled states and entanglement-assisted coding in the presence of memory. Hence, we have suggested another approach for the transmission of information by using semi-quantum dense coding. In this approach, we use an entangled pair that was previously shared between the parties, and they compromised for a transmission of quantum states through quantum channels in a specific manner. Our results show that if noise in the consecutive uses of the channels is assumed to be Markov-correlated quantum noise, then, the use of semi-quantum approach to quantum dense coding enhances classical capacity of quantum channels in various types of Pauli channels.

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